Lecture 7: Techniques of Integration

IV. Integration of Rational Functions by Partial Fractions, part I (7.5)

Which integral would you rather evaluate?

$$\int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx \quad \text{or} \quad \int \frac{x+5}{x^2+x-2}\, dx$$

The method of Partial Fractions allows us to integrate a rational function by first expressing it as a sum of simpler fractions called **Partial Fractions** that can be integrated easily.

A rational function $f(x) = \dfrac{P(x)}{Q(x)}$ is **proper** if

$$deg(P) < deg(Q).$$

If $f(x)$ is **improper**, we use **long division** to write

$$f(x) = \frac{P(x)}{Q(x)} = g(x) + \frac{R(x)}{Q(x)}, \quad deg(R) < deg(Q).$$

where g, P, R, Q are polynomials.

ex. Write $\dfrac{x^3 + x}{x - 1}$ as a proper rational expression.

$(x^2 + x + 2 + \frac{2}{x-1})$

Partial Fraction decomposition-PDF

Factor the denominator $Q(x)$ as far as possible:

Fact: Any polynomials $Q(x)$ of real coefficients can be factored as a product of linear and/or irreducible quadratic factors. This gives us 4 possible cases of decomposing a proper rational function. We start with the simplest case:

Case 1 $Q(x)$ is a product of **distinct linear factors with no repeats.**

$$Q(x) = (a_1x + b_1)\cdots(a_nx + b_n)$$

then, there is a **partial fraction decomposition (PFD)**

$$f(x) = \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_n}{a_nx + b_n}$$

for suitable constants $A_1, \cdots A_n$.

Once we have found the PFD, we can integrate the rational function $f(x)$ more easily.

<u>ex.</u> First, practicing writing out the **form** of the partial fraction **decomposition**.
(Do not determine the constants.)

1. $\dfrac{3x}{(2x+3)(x-1)}$

2. $\dfrac{x^2+1}{x^2-1}$ $\quad (1 + \frac{A}{x+1} + \frac{B}{x-1})$

Evaluating the following integrals:

ex. $$\int \frac{1}{(x+4)(x-1)}\, dx.$$

PFD:

$$\frac{1}{(x+4)(x-1)} = \frac{\quad}{x+4} + \frac{\quad}{x-1} \qquad (\star)$$

Finding the Constants:

1. Multiply both sides by $(x+4)(x-1)$ to clear denominators:
$$1 = A(x-1) + B(x+4)$$
2. Let $x = -4$ (to make the B's term disappear)
$$1 = -5A \Longrightarrow A = -\tfrac{1}{5}$$
3. Let $x = 1$ (to make the A's term disappear)
$$1 = 5B \Longrightarrow B = \tfrac{1}{5}$$

The resulting PFD is

$$\frac{1}{(x+4)(x-1)} = \frac{1}{5}\left(\frac{1}{x-1} - \frac{1}{x+4}\right)$$

Evaluate:

$$\int \frac{1}{(x+4)(x-1)}dx = \frac{1}{5}\int \left(\frac{1}{x-1} - \frac{1}{x+4}\right)\ dx.$$
$$= \frac{1}{5}(\ln|x-1| - \ln|x+4|)$$
$$= \frac{1}{5}\ln|\frac{x-1}{x+4}| + c$$

There is a quick way to determine these constants ($\star$) by using the

Cover-Up Method:

<u>To find A:</u>

Cover B's term on RHS, and $(x+4)$ on both sides, plug in $x = -4 \to A = -\frac{1}{5}$.

<u>To find B:</u>

Cover A's term on RHS, and $(x-1)$ on both sides, plug in $x = 1 \to B = \frac{1}{5}$

Recap the 4 steps in integrating a rational expression using partial fractions–

1. First, make sure the expression is **Proper**
2. **Partial Fraction Decomposition** (PFD)
3. Finding the **Contants**
4. Integrate the **Partial Fractions.**

ex. $$\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\, dy$$

PFD: $$\frac{4y^2 - 7y - 12}{y(y+2)(y-3)} =$$

Constants: (use cover-up method:)

Let $y = 0$, $\quad A =$

Let $y = -2$, $\quad B =$

Let $y = 3$, $\quad C =$

Evaluate:

$$\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)} dy = \int_1^2 \left(\frac{\quad}{y} + \frac{\quad}{y+2} + \frac{\quad}{y-3} \right) dy$$

$$\vdots$$

$= \frac{9}{5}(\ln(8/3))$.

Case 2 $Q(x)$ has **repeated linear factors**

$$Q(x) = (ax+b)^n,$$

then, there is a PFD

$$\frac{R(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

for suitable constants $A_1, \cdots A_n$.

Practicing writing out the **form** of the partial fraction decomposition for the case 2 type. (Do not determine the contants.)

3. $\dfrac{x^2+9x-12}{(3x-1)(x+6)^2}$

4. $\dfrac{1}{x^4-x^3}$ $\quad (\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1})$

ex. $\displaystyle\int \frac{1}{(x+5)^2(x-1)}dx.$

PFD: $$\frac{1}{(x+5)^2(x-1)} =$$

Constants:

Cover-Up: $x = 1,$

$x = -5,$

3rd constant **can't** be found the same way, but...
Plug in $x =$

Evaluate:

$$\int \frac{1}{(x+5)^2(x-1)}dx = \int \qquad dx.$$

$=$

$= \frac{1}{36} \ln |\frac{x-1}{x+5}| + \frac{1}{6}(\frac{1}{x+5}) + c$

NYTI: $\displaystyle\int \frac{ds}{s^2(s-1)^2}$

PFD: $\displaystyle\frac{1}{s^2(s-1)^2} =$

Contants:

Cover-Up: $s = 0,$

$s = 1,$

To find the other coefficients:

Evaluate:

$$\int \frac{ds}{s^2(s-1)^2} = \int \qquad ds.$$

$=$

$= 2\ln|\frac{s}{s-1}| - \frac{1}{s} - \frac{1}{s-1} + c$